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A SHORT SURVEY: THE JACOBIAN CONJECTURE AND FREE LIE ALGEBRAS

NAIME EKICI

Abstract. This a brief survey on the Jacobian conjecture for free Lie algebras. We present the main results of O.H, Keller., A.A. Mikhalev and V. Shpilrain.In particular we give some applications of the Jacobian conjecture. Some of these applications yield nice criterions for recognizing automorphisms of relatively free Lie algebras and free color Lie superalgebras. We also give examples of non-tame automorphisms of certain free Lie algebras.

1. Introduction

In this brief survey, we mention some nice reformulations of the Jacobian conjecture for free Lie algebras over a field of characteristic zero. This conjecture is very interesting, notorious and fun. Recently, we have a number of remarkable results about the Jacobian conjecture. Collecting these results would take several books. So, we will focus here only on free Lie algebras and we give an overview of the conjecture.

The Jacobian conjecture was formulated by O.H, Keller in 1936 [7] . The author has presented the conjecture as a problem associated with Ganze Cremona transformations. Many equivalent forms of the conjecture were presented by several authors.

Our main idea is to explain how automorphisms of a free Lie algebra and the Jacobian conjecture are connected.

In [2], Birman has given a matrix characterization of automorphisms of a free group. Some results in algebra and Birman's "inverse function theorem" have led to important applications of non-commutative Jacobian matrix.

About 20 years later Shpilrain [16] has proved the analog of Birman's "inverse function theorem" for free Lie algebras. He gave a criterion "for n elements of a free Lie algebra of rank n to be a generating set" [16]. In 1993, U.U, Umirbaev [21] generalized Birman's theorem to relatively free Lie algebras.

In [11], A.A, Mikhalev, V. Shpilrain and A.A, Zolotykh showed that the rank of the subalgebra of a free Lie algebra generated by a finite set of elements, is equal to the maximal number of left independent rows of the corresponding Jacobian

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matrix. This result shows that the rows of the Jacobian matrix are responsible for the rank of the corresponding set of a free Lie algebra.

The following remarkable situations with the Jacobian matrix of a finite set of elements are observed in [11]:

(1) Invertibility of the Jacobian matrix means that a given set of elements is a free generating set or a part of a free generating set.
(2) The maximal number of independent rows of the Jacobian matrix equal to the rank of the subalgebra generated by a given set of elements.
We give here one more situation:
(3) The invertibility of the Jacobian matrix means that a given endomorphism is an automorphism.

In section 3 we investigate the works done by Aydın, Esmerligil, Ekici, Kıral, Temizyürek and Özkurt. We survey their contributions.

We now give some nice re-formulations of the Jacobian Conjecture:

2. THE JACOBIAN CONJECTURE

In this section we give three re-formulations of the Jacobian Conjecture. We use different notation in different sections.

2.1. **O.H.Keller's Formulation.** Let K be a field of characteristic zero, $X = \{x_1, x_2, \ldots, x_n\}$ and $K[X]$ be the polynomial algebra in n variables. For any endomorphism φ of the algebra $K[X]$, we define the Jacobian matrix $J(\varphi)$ as

$$J(\varphi) = \left(\frac{\partial\varphi(x_i)}{\partial x_j}\right), 1 \leq i, j \leq n.$$

Theorem 2.1. *(O.H.Keller [7]) Let φ be an endomorphism of $K[X]$. lf the Jacobian matrix $J(\varphi)$is invertible , then φ is an automorphism of $K[X]$.*

The Jacobian conjecture is obviously true for $n = 1$. For $n \geq 2$ it is still open. In the case $n = 2$ we have a nice result due to Formanek [5] : Let φ be an endomorphism of $K[x_1, x_2]$. If $\varphi(K[x_1, x_2])$ contains a coordinate polynomial (that is, a polynomial which is included in a generating set of cardinality 2 of the algebra $K[x_1, x_2]$) then φ is an automorphism.

This result is weaker than the Jacobian conjucture since, it asks for $\varphi(K[x_1, x_2])$ contains a coordinate polynomial.

2.2. **V.Shpilrain's Formulation.** Let F be the free Lie algebra generated by a finite set $\{x_1, x_2, \ldots, x_n\}$, $n \geq 2$, over a field of characteristic zero. For an ideal R of F we denote by R^2 the second derived subalgebra of F. Let $U(F)$ be the universal enveloping algebra of F and φ be the natural homomorphism of F onto F/R. By $d_i(\nu)$ we denote the i th Fox derivative of an element ν of $U(F)$[6].

Theorem 2.2. *(V.Shpilrain [16]) Let R be an ideal of $F$and let $y_1, y_2, \ldots, y_n$ be elements of F. Then the Lie algebra F/R^2 is generated by the images $\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n)$ if and only if the Jacobian matrix $[\frac{\partial\varphi(y_i)}{\partial x_j}]_{1\leq i,j\leq n}$ has a left inverse over $U(F/R)$.*

Shpilrain's theorem leads to the following corollary.

Corollary 2.3. *(Shpilrain [16]) A mapping $\theta : x_i \to y_i$,$1 \leq i \leq n$, induces an automorphism of F if and only if the Jacobian matrix $\left(\frac{\partial y_i}{\partial_{x_j}}\right)_{1\leq i,\gamma\leq n}$ has a left inverse over $\cup(F)$.*

The assertion of this corollary gives us a criterion for n elements of a free Lie algebra of rank n to be a generating set, which is an analog of Birman's "inverse function theorem". A smilar result of this corollary is obtained by C. Reutenauer [14] and U.U.Umirbaev[21].

2.3. **U.U.Umirbaev's Formulation.** Let L be the free Lie algebra with free generators $x_1, ..., x_n$. The partial derivative of an element f with respect to x_i denoted by $\frac{\partial f}{\partial x_i}$, $1 \leq i \leq n$. Let T be an ideal of L. The lower central series

$$T = T_1 \supseteq T_2 \supseteq ... \supseteq T_n \supseteq ...$$

is defined inductively by

$$T_2 = [T, T], T_{n+1} = [T_n, T], n \geq 1.$$

Let $\widehat{L} = L/T, \overline{L} = L/T_2, \widetilde{L} = L/T_{c+1}, c \geq 1$. We denote the images of an element $f \in L$ (or $f \in \cup(L)$) under natural homomorphisms as following: By $\overline{f}$ in $\overline{L}$ (or in $U(\overline{L})$), by $\widehat{f}$ in $\widehat{L}$ (or in $U(\widehat{L})$), and by $\widetilde{f}$ in $\widetilde{L}$ (or in $U(\widetilde{L})$).

The column vector $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}\right)^t$ is denoted by $\partial(f)$, where $f \in L$ and t indicates transposition.

In [21] U.U.Umirbaev has defined partial derivatives for extensions of nilpotent Lie algebras and proved an analog of the *Corollary 2.3* for free algebras in the varieties $N_cM, c \geq 1$, where M stands for an arbitrary homogeneous variety of Lie algebras and N_c is the variety of all nilpotent Lie algebras of class $\leq c+1$.

Theorem 2.4. *(U.U.Umirbaev[21]) The endomorphism $\varphi : \widetilde{L} \to \widetilde{L}$ of the free algebra L of the variety $N_cM, c \geq 1$ is an automorphism if and only if the Jacobian matrix $J(\varphi)$ is invertible over $U(\widehat{L})$.*

This theorem leads to a criterion of invertibility for the endomorphisms of free polynilpotent Lie algebras. Umirbaev also generalized Birman's result to arbitrary primitive systems of elements in a finitely generated free algebra of the variety N_cU, where U stands for the abelian variety.

Theorem 2.5. *(U.U.Umirbaev[21]) The following conditions one equivalent:*

(1) The system of elements $\widetilde{f_1}, \widetilde{f_2}, ..., \widetilde{f_k}$ of the algebra $\widetilde{L}$ is primitive;
(2) The matrix $\left(\widetilde{\partial}\left(\widetilde{f_1}\right), ..., \widetilde{\partial}\left(\widetilde{f_k}\right)\right)$ is left invertible;
(3) The minors of order k of the matrix $\left(\widetilde{\partial}\left(\widetilde{f_1}\right), ..., \widetilde{\partial}\left(\widetilde{f_k}\right)\right)$ generate the unitary ideal of the algebra $U\left(\widetilde{L}\right)$.

Above theorem is analogous to the result by V.A.Roman'kov [15]and E.I.Timoshenko [20]. The Jacobian conjecture for free color Lie superalgebras and Lie p-superalgebras over fields,for free Lie algebras over rings,for free commutative algebras was proved in [9], [12] and [22] respectively.

The Jacobian conjecture has inspired research on some problems such as recognizing automorphisms in free groups and free algebras.

3. Some Applications

3.1. **The Commutator Test.** In this section we give some results in free Lie algebras of finite rank and in relatively free Lie algebras of rank two which are based on the Jacobian Conjecture.

Let F be the free Lie algebra generated by a set $\{x, y\}$ over a field K of characteristic zero and R be an ideal of F. In [18] A. Temizyürek and N. Ekici considered the free Lie algebra $F/[R', F]$ and gave a necessary and sufficient condition for any subset $\{u, v\}$ of $F/[R', F]$ to be a generating set. By $U(F)$ we denote the universal enveloping algebra of F.

Theorem 3.1. *(A.Temizyürek,N.Ekici [18]) Let u,v $\in F$ and $0 \neq \alpha \in K$. If*
$$[u, v] \equiv \alpha [x, y] \,(mod\, [R', F]) \ ,$$
then the set $\{u, v\}$ generates F modulo $[R', F]$.

Theorem 3.2. *(A.Temizyürek,N.Ekici [18]) Let R be an ideal of F such that $R \subseteq F'$ and $U(F/R)$ is an integral domain. If an endomorphism φ of F induces an automorphism of the algebra $F/[R', F]$ then for any $0 \neq \alpha \in K$,*
$$[\varphi(x), \varphi(y)] \equiv \alpha [x, y] \,(mod\, [R', F]).$$

This criterion known as the commutator test.The commutator test does not in general holds for a large class of relatively free Lie algebras. In [3] Z. Esmerligil and N. Ekici have considered the validity of the commutator test for the free Lie algebras of the form F/R'.

Theorem 3.3. *(Z. Esmerligil, N. Ekici [3]) Let $R \subset F'$ and $U(F/R)$ be an ore domain. Then there exist elements $u, v \in F$ such that $[u, v] = \alpha [x, y]$, $0 \neq \alpha \in K$, but the endomorphism θ defined by*
$$\theta : x \to u \ , \ y \to v$$
does not induces an automorphism of the algebra F/R'.

3.2. **Non-Tameness of Automorphisms of Certain Relatively Free Lie Algebras.** In this section we give some applications of the Jacobian conjecture with respect to non-tameness of automorphisms of certain relatively free Lie algebras.

Let L be the free Lie algebra generated by a finite set $\{x_1, ..., x_n\}$ over a field of characteristic zero, $L' = [L, L]$ and L_m be m-th lower central term of L.

In [13] Z. Özkurt and N. Ekici proved that the free metabelian Lie algebra L/L'' and the free Lie algebra $L/(L_m)'$ have non-tame automorphisms.

Theorem 3.4. *(Z. Özkurt, N. Ekici [13]) The endomorphism φ defined as*
$\varphi : x_1 \to x_1 + [[x_1, [x_{j_1}, x_{j_2}]], x_{j_3}]$, $x_i \to x_i$, $i \neq 2$, $j_\alpha \neq j_\beta$, $1 \leq \alpha, \beta \leq 3$, $j_\gamma \neq 1$, $\gamma = 1, 2$,
is a non-tame automorphism of the free metabelian Lie algebra L/L''.

Theorem 3.5. *(Z. Özkurt, N. Ekici [13]) The endomorphism φ defined as*
$$\varphi : x_1 \to x_1 + [x_1, v] \ , \ x_i \to x_i \ , \ i \neq 1,$$
of L induces a non-tame automorphism of the algebra $L/(L_m)'$, where $v = [[..., [x_{j_1}, x_{j_2}], ...], x_{j_m}]$, $j_k \neq 1$, $k = 1, ..., m$, $m \geq 3$.

Finally we give two applications of the Jacobian conjecture in free color Lie superalgebras.

3.3. **Applications in Free Color Lie Superalgebras.** Let L be a free color Lie super algebra generated by a set $\{x, y\}$ over a field K of characteristic zero , G be an abelian group and $A(X)$ be the free G-graded associative algebra over K.

In the following theorem E. Aydın and N. Ekici obtained a criterion (look like to the commutator test) for a given subset of L to be a generating set.

Theorem 3.6. *(E.Aydın, N. Ekici* [1]*) Let h_1, h_2 be G-homogeneous elements of L and H the subalgebra they generate . Then $H = L$ if and only if*

$$[h_1, h_2] = \alpha\,[x, y] + \beta\,[[x, x]] + \gamma\,[y, y]\,,\ \textit{where}\ \alpha, \beta, \gamma \in K\,\{0\}\,.$$

In [8] E. Kıral and N. Ekici considered the the double Jacobian matrix for recognizing automorphims of free color Lie superalgebras of finite rank $n \geq 2$. They proved an analog of Shpilrain's [17] result for automorphisms of a free color Lie superalgebra.

Let $X = \{x_1, ..., x_n\}$ be a G-graded set and F is the free color Lie superalgebra with the set X of free generators. We define for any element $u \in F$ the double Jacobian matrix $J_D(u)$ as

$$J_D(u) = \left(\left(\frac{\partial u}{\partial x_j}\right)\frac{\partial}{x_i}\right)_{1\leq i,j\leq n}\,,$$

where $\frac{a\partial}{\partial x_i}$ is the right Fox derivation and $\frac{\partial a}{\partial x_i}$ is the left Fox derivation.

Theorem 3.7. *(E. Kıral, N. Ekici* [8]*) Let ψ be an endomorphism of the free color Lie superalgebra F. Then it is an automorphism:*

a) If and only if the double Jacobian matrix $J_D(\psi(u))$ is invertible over $A(X)$ with $u = [x_1, x_2] + ... + [x_{n-1}, x_n]$, n even, $d(\psi(x_i)) = d(x_i)$, $1 \leq i \leq n$.

b) If and only if the double Jacobian matrix $J_D(\psi(v))$ is invertible over $A(X)$ with $v = [x_1, x_1] + ... + [x_n, x_n]\,,\ d(\psi(x_i)) = d(x_i) \in G_-$, $1 \leq i \leq n$.

4. **Concluding Remarks**

The Jacobian conjecture does not in general hold for a free associative algebra.There are simple examples of invertible Jacobian matrices which do not correspond to an automorphism of a free associative algebra. However, there are some partial results in the literature mentioning the connection between automorphisms of free associative algebras and the Jacobian Conjecture. For surveys of recent progresses in free groups, polynomial algebras, free associative algebras and free color Lie superalgebras see [10] and [4].

References

[1] E.Aydın, N. Ekici, On generators of free color Lie superalgebras of rank two, J. Lie Theory, 12,2, 529-534,2002.

[2] J.S Birman , An inverse function theorem for free groups, Proc Amer ,Math.soc.41,634-638,1973.

[3] Z. Esmerligil, N. Ekici, Commutator test for the Lie algebras $F/R'$of rank two, XIX th National Math. Conf., Conference book, 685-690, Dumlupınar Univ., Kütahya, Turkey,2006.

[4] A.Van dan Essen, Seven lectures on polynomial automorphisms.In "A.van dan Essen (*Ed.*) ,Automorphisms of Affine spaces" Kluwer Academic Publisher,Dordrecht,3-39,1995.

[5] E,Formanek,observations about the Jacobian conjecture,Houston J.Math.20,369-380,1994.

[6] R.H.Fox, Free differential calculus,I.derivation in free group ring,Ann. of Math.2,57,547-560,1953.

[7] O.Keller,Ganze Cremona-Transformationen, Monatsh. Math. Phys.47,299-306,1939.

[8] E.Kıral, N. Ekici, Recognizing automorphisms of free color Lie superalgebras, J. Algebra Appl.,1, 3, 335-340, 2002.
[9] A.A.Mikhalev,On right ideals of free associative algebras generated by free color Lie p-superalgebras,Uspekhi.Mat.Nauk.47,no.5,187-189,1992
[10] A.A.Mikhalev, V.Shpilrain, Jie-Tai Yu, Combinatorial problems about free groups and algebras ,(survey).
[11] A.A. Mikhalev, V.Shpilrain,A.A. Zolotykh, Subalgebras of free algebras , Proc.Amer .Math.Soc.124,1977-1984,1996.
[12] A.A. Mikhalev,A.A. Zolotykh, An inverse function theorem for free Lie algebras over commutative rings , Algebra Colloquium ,2,No.3,213-220,1995.
[13] Z.Özkurt, N. Ekici, An application of the Dieudonne determinant :Detecting non-tame automorphisms, J. Lie Theory, 18,1, 205-214, 2008.
[14] C.Reutanauer ,Applications of a non commutative Jacobian matrix ,J.Pure Appl.Alg.,77,169-181,1992.
[15] V.A.Roman'kov, A criterion of primitiveness for a system of elements in a free metabelian group,Ukrain.Mat.Zh.,43,No.7,996-1002,1991.
[16] V.Shpilrain,On generators of L/R^2 Lie algebras,Proc.Amer.Math.Soc.,119,No.4,1039-1043,1993.
[17] V.Shpilrain, Test elements for endomorphisms of free groups and algebras, Israel J.Math., 92, 307-316, 1995.
[18] A.Temizyürek, N.Ekici, Commutator test for the Lie algebras $F/[R', F]$, XIX th National Math. Conf., Conference book, 109-115, Dumlupınar Univ., Kütahya, Turkey,2006.
[19] A.Temizyürek, N.Ekici, A particular test element of a free solvable Lie algebra of rank two, The Rocky Mountain J. Math.,37, 4, 1315-1326, 2007.
[20] E.I.Timoshenko, On inclusion of given elements in a basis for a free metabelian group,Viniti,2699,1988.
[21] V.V.Umirbaev,Partial derivatives and endomorphisms of some relatively free ,Lie algebras ,Sib.Math.J.,34,1161-1170,1993.
[22] A.V.Yagzhev,Endomorphisms of free algebras ,Sib.Math.J.21,No.1,133-141,1980.

Naime Ekici joined the Faculty of Science,Department of Mathematics at Çukurova University in 1981. She received his Ph. D. in 1983 from the Dept. of Mathematics at Çukurova University. She worked as a Professor in Mathematics Department between 1997-2016.
Her recent works have focused on the Non-associative Algebras, Lie Algebras, Parafree Lie Algebras. She served as an advisor for 12 Ms. and 10 Ph.D. graduate student. She was assigned as Mathematics Department Head between 2007 and 2010 and also as Project Unit Support (BAP) Member between 2012-2016. She has more than 30 scientific studies and many proceeding papers.
In 2016, she retired as a Professor and still continues her scientific studies.
She is married and has two sons.

Çukurova University, Adana, Turkey.
E-mail address: nekici@cu.edu.tr