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EXISTENCE AND UNIQUENESS RESULTS OF FUZZY FRACTIONAL DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

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Abstract. In this paper, we discus the existence and uniqueness of mild solution to the fuzzy Cauchy Problem for the fractional differential equation with nonlocal $D^q x(t) = Ax(t) + t^n f(t, x(t), Bx(t)), \quad t \in [0,T], n \in \mathbb{Z}^+, \quad x(0) = x_0 + g(x)$ where $0 < q < 1$, A is the generator of the fuzzy strongly semigroup $(S(t))_{t\geq 0}$ on E^n.

This is an example of a special section head

1. Introduction

The origin of Fractional Calculus goes back to Newton and Leibniz in the seventieth century. It is a generalization of ordinary differential equations and integration to arbitrary non integer orders. Fractional Calculus is widely and efficiently used to describe many phenomena arising in Engineering, Physics, Economy, and Science. Recently, fractional differential equations have attracted many authors (see for instance [111713, 151720] and references therein).

The following equation

$$\begin{cases} D^q_{0+}x(t) = f(t, x(t)), \quad 0 < t < 1 \\ x(0) + x'(0) = 0, x(1) + x'(1) = 0 \end{cases}$$

where D^q_{0+} denotes the Caputo fractional derivative with $1 < q \leq 2$ was studied by S. Zhang [20] and the existence of positive solutions was obtained using classical fixed point theorems.

In [19], the author studied both the local and global existence of solutions to the equation

$$\begin{cases} D^q x(t) = f(t, x(t)), \\ x^{(k)}(t_0) = x_0^{(k)}, \quad k = 0, 1, ..., n-1 \end{cases}$$

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in a finite dimensional space. The results are obtained via construction and the contraction mapping principle. Recently G.M. NGurkata [17] has considered the Cauchy Problem with nonlocal conditions

$$\begin{cases} D^q x(t) = f(t, x(t)), & t \in [0, T], \\ x(0) + g(x) = x_0, \end{cases}$$

in a general Banach space X with $0 < q < 1$. By means of the Krasnoselskiis Theorem, existence of solution was also obtained. In his pioneering paper [4], K. Deng has indicated that the nonlocal condition $x(0) + g(x) = x_0$ can be applied in physics with better effect than the usual local Cauchy Problem $x(0) = x_0$. Deng used

$$g(x) = \sum_{k=1}^{p} c_k x(t_k),$$

where $c_k, k = 1, 2, ..., p$, are given constants and $0 < t1 < t2 < ... < t_p.T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, the Cauchy problem allows additional measurements at $t_k, k = 1, 2, ..., p$. From a theoretical stand point, the nonlocal condition above appears more general than the classical initial value problem.

Lets observe also that since Dengs paper, such problem has also attracted several authors including A. Aizicovici, L. Byszewski, K. Ezzinbi, Z. Fan, J. Liu, J. Liang, Y. Lin, T.-J. Xiao, E. Hernndez, H. Lee, etc. (see for instance [1178, 14, 17] and the references therein).

We are motivated here by [9] where the authors study the existence and uniqueness of the mild solution to the problem with initial value

$$\begin{cases} D^q x(t) = f(t, x(t)), Gx(t), Sx(t), & t > t_0 \\ x(0) = x_0, \end{cases}$$

where $0 < q \leq 1$, and A is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$.

In this regard G.M.Mophou and G.M.N'Gurkata [see] are study the existence of the mild solution for fractional differential equations with nonlocal conditions:

$$\begin{cases} D^q x(t) = Ax(t) + t^n f(t, x(t), Bx(t)), & t \in [0, T], n \in \mathbb{Z}^+, \\ x(0) = x_0 + g(x), \end{cases}$$

Where T is a positive real, $0 < q < 1$, A is the generator of a C^0-semigroup $(S(t))_{t \leq 0}$ on a Banach space $X, Bx(t) := \int_0^t K(t, s)x(s)ds, K \in C(D, \mathbb{R}^+)$ with $D := t\{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ and

$$B^* = \sup_{t \in [0,T]} \int_0^t K(t, s)ds < \infty$$

$f : \mathbb{R} \times X \times X \to X$ is a nonlinear function, $g : C([0, T], X) \to D(A)$ is continuous and $0 < q < 1$. The derivative D^q is understood here in the Caputo sense.

In this paper we are concerned with the existence and uniqueness of the mild solution to fuzzy Cauchy Problem for the fractional differential equation with non-local conditions

$$\begin{cases} D^q x(t) = Ax(t) + t^n f(t, x(t), Bx(t)) \quad t \in [0,T] \quad n \in Z \\ x(0) = x_0 + g(x) \end{cases} \tag{1.1}$$

where T is a positive real, $0 < q < 1$, A is the generator of the fuzzy strongly semigroup $(S(t))_{t\geq 0}$ on E^n, $B(x(t)) = \int_0^t K(t,s)x(s)ds, \quad K \in C(D, \mathbb{R}^+)$
$D = \{(t,s) \in \mathbb{R}^2 + \quad 0 \leq s \leq t \leq T\}$
$B^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds$
$f : [0,T] \times E^n \times E^n \to E^n$ is a nonlinear function.
$g : C([0,T], E^n) \to D(A)$ is continuous and $0 < q < 1$ the fuzzy derivative: D^q is understood in the caputo since.

1.1. **Existence and uniqueness.**

$$\begin{cases} D^q x(t) = Ax(t) + t^n f(t, x(t), Bx(t)) \quad t \in [0,T] \quad n \in Z \\ x(0) = x_0 + g(x) \end{cases} \tag{1.2}$$

where T is a positive real, $0 < q < 1$, A is the generator of the fuzzy strongly semigroup $(S(t))_{t\geq 0}$ on E^n, $B(x(t)) = \int_0^t K(t,s)x(s)ds, \quad K \in C(D, \mathbb{R}^+)$
$D = \{(t,s) \in \mathbb{R}^2 + \quad 0 \leq s \leq t \leq T\}$
$B^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds$
$f : [0,T] \times E^n \times E^n \to E^n$ is a nonlinear function.
$g : C([0,T], E^n) \to D(A)$ is continuous and $0 < q < 1$ the fuzzy derivative: D^q is understood in the caputo since.

Definition 1.1. we say that x is a mild solution of 1.2 if:

(1) $x \in C([0,T], E^n), x(t) \in D(A)$ for all $t \in [0,T]$
(2) $x(t) = S(t)(x_0 + g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{q-1} s^n S(t-s) f(s, x(s), Bx(s))ds$

We assume that:

- $(H_1) f : [0,T] \times E^n \times E^n \to E^n$ is continuous.
- (H_2) there exist functions $u_1, u_2 \in L^1_{loc}([0,T], \mathbb{R}^+)$ such that: $D(f(t,x,u), f(t,y,v)) \leq u_1 D(x,y) + u_2 D(u,v)$ for all $t \in [0,T]$ and $x, y, u, v \in E^n$
- (H_3):$g : C([0,T], E^n) \to D(A)$ is continuous and there exists a constant b nonegative such that $D(g(x), g(y)) \leq bD(x,y)$ for all $x, y \in C_T$
- (H_4):$x_0 \in D(A)$
- (H_5): the function $\varepsilon_n(t) : [0,T] \to R^+$, $n \in Z$

$$\varepsilon_n(t) = M_T[b + \frac{T^{q-1}}{\Gamma(q)} \frac{t^{n+1}}{n+1}(\| \mu_1 \|_{L^1_{loc}} + B^* \| \mu_2 \|_{L^1_{loc}})]$$

satisfied $0 < \varepsilon_n(t) \leq \gamma < 1$ for all $0 \leq t \leq T$.

Theorem 1.2.

Under assumption (H_1)-(H_5), if A is the generator of a fuzzy strongly semigroup $(S(t))_{t\geq 0}$ on E^n, then the problem 1.2 has a unique mild solution.

Proof.
Define $F : C \to C$ by:

$$x(t) \mapsto F(x(t)) = S(t)(x_0 + g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) s^n f(s, x(s), Bx(s)) ds$$

Step 1

Let $x \in C$ and $h \in \mathbb{R}^+$

$$\begin{aligned}
D(F(x(t+h)), F(x(t))) &= D(S(t+h)(x_0+g(x)) + \frac{1}{\Gamma(q)} \int_0^{t+h} (t+h-s)^{q-1} s^n S(t+h-s) f(s, x(s), Bx(s))d \\
&+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n S(t-s) f(s, x(s), Bx(s)) ds) \\
&\leq D(S(t)S(h)(x_0+g(x)), S(t)(x_0+g(x))) \\
&+ \frac{1}{\Gamma(q)} D(\int_0^h (t+h-s)^{q-1} s^n S(t+h-s) f(s, x(s), Bx(s)) ds + \int_h^{t+h} (t+h-s)^{q-1} s \\
&\leq D(S(t)S(h)(x_0+g(x)), S(t)(x_0+g(x))) \\
&+ \frac{1}{\Gamma(q)} D(\int_0^h (t+h-s)^{q-1} s^n S(t+h-s) f(s, x(s), Bx(s)) ds, \widetilde{0}) \\
&+ \frac{1}{\Gamma(q)} D(\int_0^t (t-s)^{q-1} (s+h)^n S(t-s) f(s+h, x(s+h), Bx(s+h)) ds, \int_0^t (t-s)^{q-1} \\
&\leq D(S(t)S(h)(x_0+g(x)), S(t)(x_0+g(x))) \\
&+ \frac{1}{\Gamma(q)} \int_0^h D((t+h-s)^{q-1} s^n S(t+h-s) f(s, x(s), Bx(s)), \widetilde{0}) ds \\
&+ \frac{1}{\Gamma(q)} M T^{q-1} \int_0^t \exp^{\omega(t-s)} D((s+h)^n f(s+h, x(s+h), Bx(s+h)), s^n S(t-s) f(s,
\end{aligned}$$

it is clear that $D(S(t)S(h)(x_0+g(x)), S(t)(x_0+g(x))) \to 0 \quad \text{as} \quad h \to 0$

and $\int_0^h D((t+h-s)^{q-1} s^n S(t+h-s) f(s, x(s), Bx(s)), \widetilde{0}) ds \to 0 \quad \text{as} \quad h \to 0$

and by the dominated convergence theorem:

$\int_0^t \exp^{\omega(t-s)} D((s+h)^n f(s+h, x(s+h), Bx(s+h)), s^n S(t-s) f(s, x(s), Bx(s)))ds \to$ 0 as $h \to 0$

Hence $F(x) \in C$ i.e F maps C into itself.

Step2:

let $t \in [0,T]$ and $x, y \in C$,then we have

$$
\begin{aligned}
D(F(x(t)), F(y(t))) &= D(S(t)(x_0+g(x)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} s^n S(t-s) f(s, x(s), Bx(s))ds, S(t)(x_0+g(y))\\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} s^n S(t-s) f(s, y(s), By(s))ds)\\
&= D(S(t)(x_0+g(x)), S(t)(x_0+g(y)))\\
&+ \frac{1}{\Gamma(q)} D(\int_0^t (t-s)^{q-1} s^n S(t-s) f(s, x(s), Bx(s))ds, \int_0^t (t-s)^{q-1} s^n S(t-s) f(s, y(s), B\\
&\leq M\exp(\omega t) D(g(x), g(y))\\
&+ \frac{1}{\Gamma(q)}\int_0^t D((t-s)^{q-1} s^n S(t-s) f(s, x(s), Bx(s)), (t-s)^{q-1} s^n S(t-s) f(s, y(s), By(s))\\
&\leq M_T b H(x,y) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} M\exp(\omega(t-s)) s^n D(f(s, x(s), Bx(s)), f(s, y(s), By(\\
&\leq M_T b H(x,y) + \frac{T^{q-1} M_T}{\Gamma(q)}\int_0^t s^n(\mu_1(s) D(x(s), y(s)) + \mu_2(s) D(Bx(s), By(s)))ds\\
&\leq M_T b H(x,y) + \frac{T^{q-1} M_T}{\Gamma(q)}(\int_0^t s^n \mu_1(s) ds) H(x,y)\\
&+ \frac{T^{q-1} M_T}{\Gamma(q)} B^*(\int_0^t s^n \mu_2(s) ds) H(x,y)\\
&\leq M_T[b + \frac{T^{q-1}}{\Gamma(q)}\frac{t^{n+1}}{n+1}(\| \mu_1 \|_{L^1_{loc}} + B^* \| \mu_2 \|_{L^1_{loc}})] H(x,y)\\
&\leq \varepsilon_n(t) H(x,y)
\end{aligned}
$$

so we get :$H(F_x, F(y)) \leq \gamma H(x,y)$ where $\gamma < 1$.

Hence F is the contraction. Then the problem 1.2 has a unique mild solution $x(t)$.

$$x(t) = S(t)(x_0+g(x)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} S(t-s) s^n f(s, x(s), Bx(s))ds$$

□

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